Experiment 13 – Poles and zeros in the z plane: IIR systems

Achievements in this experiment

You will be able to interpret the poles and zeros of the transfer function of discrete-time filters to visualize frequency responses graphically at a glance, without math. You will be able to use this knowledge to intuitively design recursive/IIR discrete-time responses.

Preliminary discussion

In a previous lab, Lab 9, we discovered how poles and zeros can be used as an intuitive tool for analyzing and designing continuous-time (CT) filters. Next, in Lab 12 we examined discrete-time (DT) FIR filters and found the same ideas could be applied there. The complex "s" plane was replaced with the complex "z" plane, and the unit circle used instead of the j axis for the representation of frequency. Because zeros only are involved in FIR filter work, this provided a convenient gateway to getting started with z-plane ideas.

In this Lab we will investigate more general DT filters that are characterized with both poles and zeros. These filters are known as *recursive* since they use feedback, and also as *Infinite Impulse Response (IIR)*. With feedback we will be able to realize much higher selectivity than possible with a comparable complexity FIR implementation. The most conspicuous example is the second-order resonator, which will open the way to achieving realistic bandpass responses. As we proceed, we will find many parallels with the CT(continuous time) filter experiments in Lab 11.

Part 1: we examine the behaviour of the basic second-order resonator implemented without zeros.

Part 2: zeros are introduced to generate lowpass, bandpass, highpass and allpass responses using the Direct Form 2 structure.

Pre-requisite work:

This preparation extends the theory covered in Lab 14 to include poles.

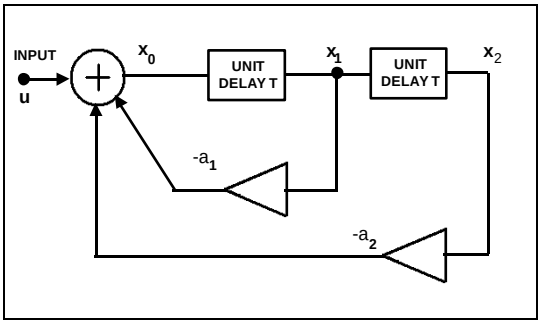


Figure 1: schematic of 2nd-order feedback structure without feedforward.

Question 1
Consider the feedback system in Figure 1.

Show that the difference equation relating the adder output x0(nT) and the input u(nT) is

$x_0(nT) = u[nT] - a_1.x_0[(n - 1)T] - a_2.x_0[(n - 2)T]$ (Eqn 1),

where nT are the discrete time points, T sec denoting the unit delay, i.e. the time between samples.

Show by substitution that e^{jnTw} is a solution, i.e. show that when $x_0(nT)$ is of the form e^{jnTw}, the input u(nT) is e^{jnTw}, multiplied by a constant (complex-valued); w is the frequency of the input in radians/sec; (the use of complex exponentials for the representation of sinusoidal signals is discussed in Lab 8, 10 and 13.

From the above, with input $u(nT) = e^{jnTw}$ obtain

$x_0/u = 1/[\ 1 + a_1.\ e^{-jTw} + a_2.\ e^{-j2Tw}]$ (Eqn 2).

Note that x_0/u is not a function of the time index n.

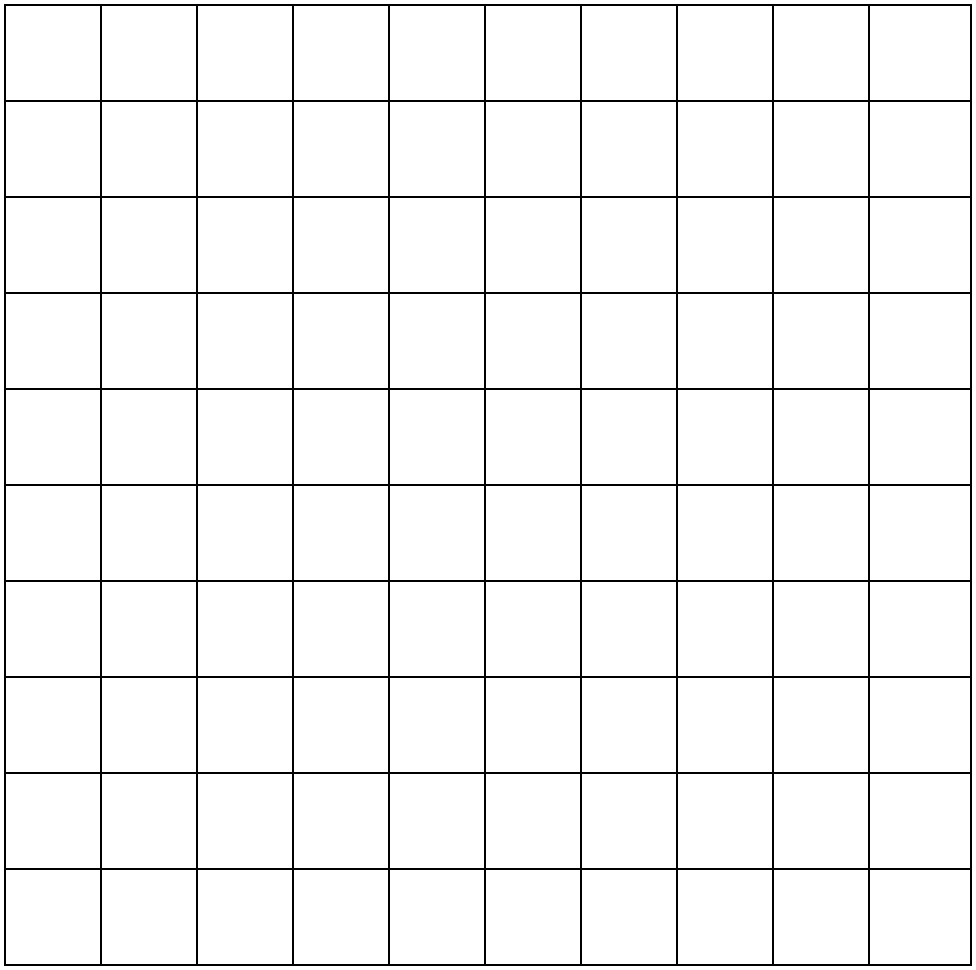
Question 2

Use this result to obtain a general expression for $|x_0/u|$ as a function of w.

Tip: to simplify the math, operate on u/x_0 instead of x_0/u, expressing the result in polar notation.

Set T = 1 sec for the time being, and plot the result for the case $a_1 = -1.6$, $a_2 = 0.902$ over the range w = 0 to π rad/sec. Label the frequency axis in Hz and in rad/sec. You should find there is a peak in the response near 0.09Hz.

Pre-lab preparation work



Graph 1:response plot

Question 3

Replace "exp(jTw)" by the symbol "z" in Eqn 2. The result is

$$H_x_0(z) = x_0/u = 1/(1 + a_1.z^{-1} + a_2.z^{-2}) = z^2 / (z^2 + a_1.z + a_2) \quad \text{(Eqn 3).}$$

The quadratic $(z^2 + a_1.z + a_2)$ can be expressed in the factored form $(z - p_1)(z - p_2)$.

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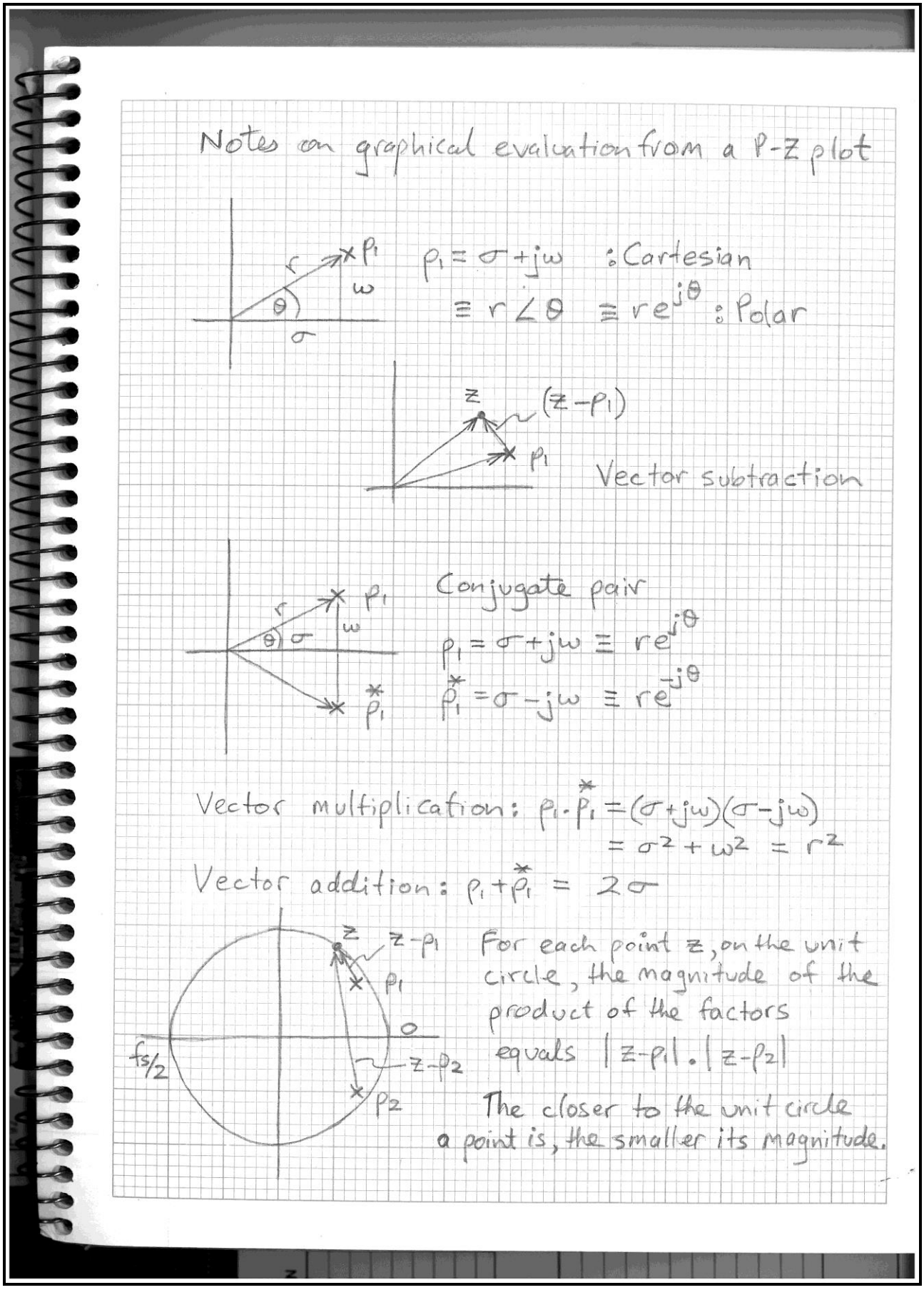


Figure 2: Notes on the graphical interpretation of pole-zero plots

Reviewing the finding of roots of the quadratic polynomial

From equation 1 above
$x_0(nT) = u[nT] - a_1.x_1(nT) - a_2.x_2(nT)$ substituting $x_1=x_0.z^{-1}$ and $x_2=x_0.z^{-1}.z^{-1}$ we arrive at
$x_0 = u - a_1x_0/z^1 - a_2x_0/z^2$
Grouping x0 terms:
$x_0(1 + a_1/z^1 + a_2/z^2) = u$

At this point we can see that although we started with negative gains in the circuit model, we now have positive values as coefficients in the quadratic equation.

Further, we arrive at:
$x_0/u = z^2/(z^2 + a_1z^1 + a_2)$ which we earlier named Eqn 3.
We now have the general quadratic form with positive coefficients.
INSIGHT: positive coefficients result in negative gains in the actual implementation

This quadratic $(z^2 + a_1z + a_2)$ can be expressed in factored form, as (z-p1)(z-p2)
Remember that z, p1 & p2 are complex numbers. You can think of these as vectors: from the origin of the z plane to a 2 dimensional point on that plane.

Each factor ie: (z-p1) and (z-p2) is a difference vector between a general point z, who's locus we restrain to the unit circle, and the 2 specific roots p1 & p2. It will be a vector, having direction and magnitude, and can be expressed in polar notation as $r\angle\theta$, or $re^{j\theta}$, or in Cartesian notation as (a + ib). Both these representation are complex numbers.

If we define p1 as (σ + iw) and its conjugate, p2 as (σ – iw) we can express the quadratic factors as:
$(z- p1)(z- *p1) = z^2 + p.*p - pz - *pz$
Switching to polar notation for convenience, $p.*p = re^{j\theta}.re^{-j\theta} = r^2$
So that leaves $z^2 + r^2 - z.(p + *p)$, and if using Cartesian notation in this instance for convenience, ie. p = σ + jw then p + *p = 2σ so
$z^2 +(-2\ \sigma)z + r^2 = z^2 -2\ \sigma\ z + r^2 = z^2 +a_1z + a_2$

Relating coefficients gives $\mathbf{a_1 = -2\sigma}$ **and** $\mathbf{a_2 = r^2}$

For stability the poles must always be inside the unit circle, hence $0 < a_2 < 1$

Changes in a_1 directly influence the real component of the pole position
a_2 has a square law relationship with r of the pole.
Other relationships, such as θ, w, imag part, can be derived from these easily with trigonometry.

The general solution for the roots of the quadratic polynomial $x_2 + a_1x + a_2$ is:
$\mathbf{x = -a_1/_2 +/- i\sqrt{[a_2-(a_1/_2)^2]}}$

With these equations in mind consider how changes in the coefficients from the math will move the poles or zeros about the unit circle, and influence the response of the system.

This lab aims to make you more familiar of the interrelationships between these parameters.

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Using the values of a_1 and a_2 given in Question 2 above, find the roots p_1 and p_2 (express the result in polar notation). Mark the position of p_1 and p_2 on the complex z plane with an "x" to indicate that they represent poles. The distance between these points and the unit circle is of key importance.

This is a parallel process to that in Lab 11 where we plotted zeros. A similar procedure was carried out in Lab 11 for a CT transfer function in the complex variable s.

Write down a formula for p_1 in terms of a_1 and a_2. Note that p_1 may be real or complex depending on a_1 and a_2. Determine the conditions for p_1 to be complex valued. For this case, express p_1 in polar notation. Take note of the fact that $|p_1|$ does not depend on a_1 (this will be useful later). Obtain p_2 from p_1.

Question 4

Satisfy yourself that the magnitude response of H_x_0 is given by

$$|H_x_0(w)| = 1/[|(e^{jTw} - p_1)|.|(exp^{jTw} - p_2)|] \qquad \text{(Eqn 4)}.$$

This provides the key for the graphical method described in Lab 13 to obtain an estimate of the magnitude response. Again, we will use T = 1 .

Plot the magnitude of the denominator for selected values of w over the range 0 to π. The quantity $|(e^{jTw} - p_1)|$ becomes quite small and changes rapidly as the point on the unit circle is moved near p_1. Plot additional points there as needed. Invert to get $|H_x_0(w)|$ and compare this with the result you obtained in (b).

Graph 2:response plot

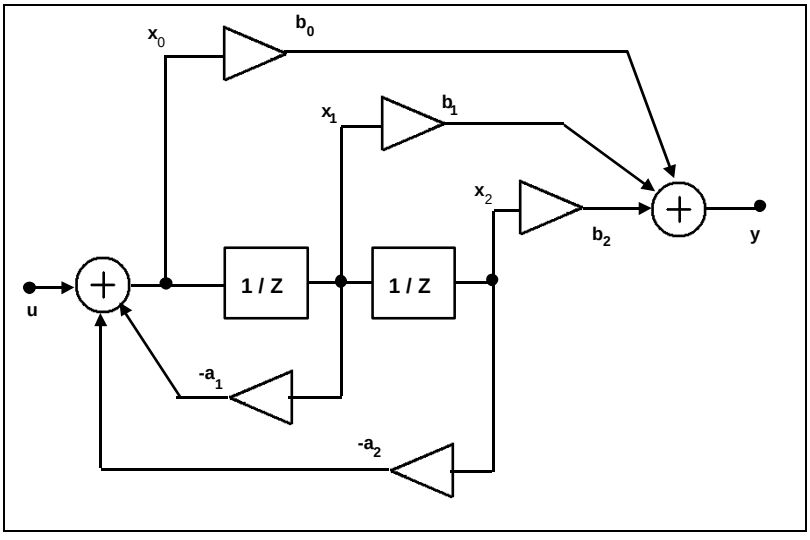


Figure 3: block diagram of 2nd-order Direct-form 2 structure with feedforward.

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Question 5

Modify Fig 1 by replacing the unit delays with a gain of 1/z and show that Eqn 3 follows by inspection using simple algebra.

Question 6

Apply this idea to show that the transfer function for the system in Fig. 3 is

$$H_y(z) = y/u = (b_0 + b_1.z^{-1} + b_2.z^{-2}) / (1 + a_1.z^{-1} + a_2.z^{-2}) \qquad \text{(Eqn5)}$$

Question 7

Use the graphical pole-zero method (covered in Experiment 14) to obtain estimates of the magnitude responses for the following cases (0 to Nyquist freq):

(i) $b_0 = b_2 = 1$, $b_1 = 2$, a_1 and a_2 as in Question 2.
(ii) $b_0 = b_2 = 1$, $b_1 = -2$, a_1 and a_2 as in Question 2
(iii) $b_0 = 1$, $b_1 = 0$, $b_2 = -1$, a_1 and a_2 as in Question 2

Which of these is lowpass, highpass, bandpass?

Graph 3:**pole-zero method**

Question 8

Consider a DT system with sampling rate 20kHz. Obtain estimates of the poles and zeros that realize a lowpass filter with cut-off near 3kHz. Obtain a highpass filter using the same poles.

Question 9

For the same sampling rate as in Question 8 obtain estimates of the poles and zeros that realize a bandpass filter centered near 3.1kHz, with 3dB bandwidth 500Hz. HINT: review Question 7

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Equipment

- PC with LabVIEW 2012 RTE and TIMS S&S SFP software installed
- Modules: AUDIO OSCILLATOR, SEQUENCE GENERATOR, UTILITIES, PC-MODULES CONTROLLER (ARB), z-TRANSFORM V2, TRIPLE ADDER V2, DIGITAL UTILITIES
- Assorted 4 mm patch leads
- TIMS-301C, or TIMS-301 with external PICO digital scope unit

Procedure

Part A – Setting up the PC-MODULES CONTROLLER and software

1. Plug the PC-MODULES CONTROLLER board into the TIMS unit.

2. Connect the PC-MODULES CONTROLLER module to the PC via the supplied USB cable and confirm that the module is detected. This is usually indicated by a brief sound from the PC.

3. Launch the TIMS S&S SFP software from the PC.

Note: To stop the TIMS S&S SFP when you've finished the experiment, it's preferable to use the STOP button on the SFP itself rather than the LabVIEW window STOP button at the top of the window. This will allow the program to conduct an orderly shutdown and close the various USB communication channels it has opened.

Experiment

Part 1: IIR without feedforward: a second-order resonator

In this part we implement and investigate the system in Fig 1.

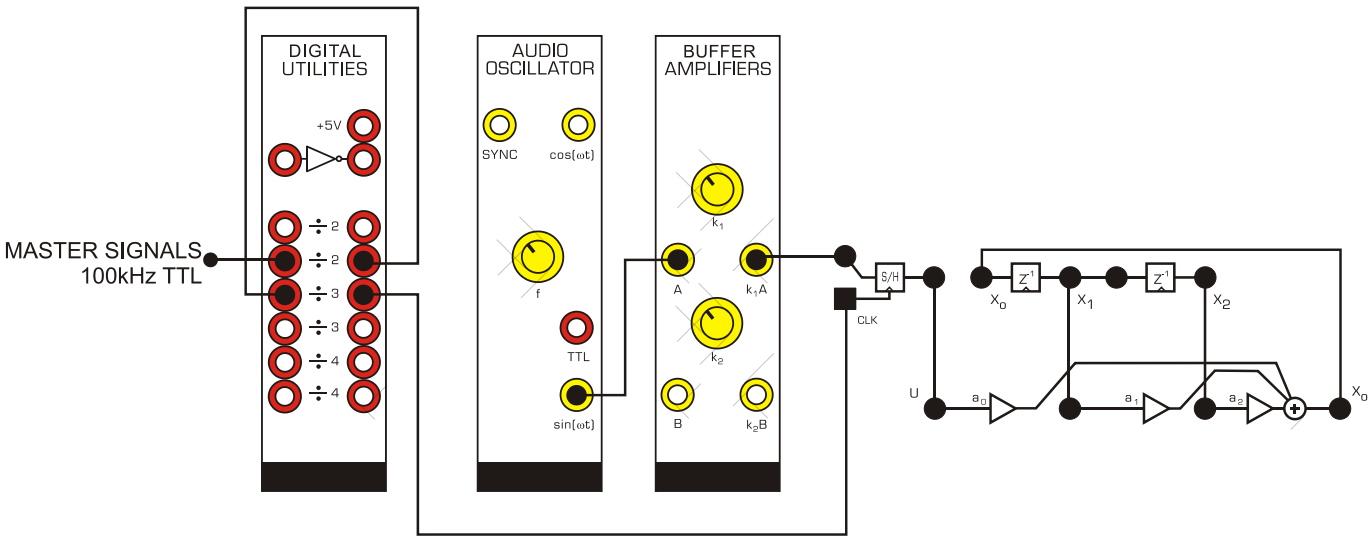


Figure 4: patching diagram for feedback structure without feedforward from Figure 1

1. Patch up a model of the system in Fig 1.

Settings are as follows:
ADDER GAINS: a0=1; a1=+1.6; a2= -0.902
CLK: 16.6kHz
AUDIO OSCILLATOR: Sinewave selected, FREQUENCY=1k; Amplitude (after BUFFERS)=1V pp

Question 10
Calculate the poles corresponding to these values. Measure and plot the magnitude response at the output of the feedback adder. Note and record the resonance frequency and the bandwidth. Use the poles to graphically predict these parameters; compare with your measurements.

Question 11
Decrease $|a_1|$ by a small amount (around 5-10%, say) and measure the effect on the resonance frequency and bandwidth. Use this to estimate the migration of the poles. Does this agree with your expectations?

Question 12
Repeat step 3 for a 5% decrease of a_2. Compare the effects of varying a_1 and a_2. Which of these controls would you use to tune the resonance frequency? Use the formulas you obtained in the preparation to explain this.

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NOTE about TAB " PZ plot" on the SFP.
This panel calculates the poles and zeros relating to the currently set ADDER gains which relate to the coefficients of the transfer function. Use this visualization tool to confirm your understanding . You may wish to move back and forth between the experiment TAB and the "PZ plot" TAB as required during the experiment.

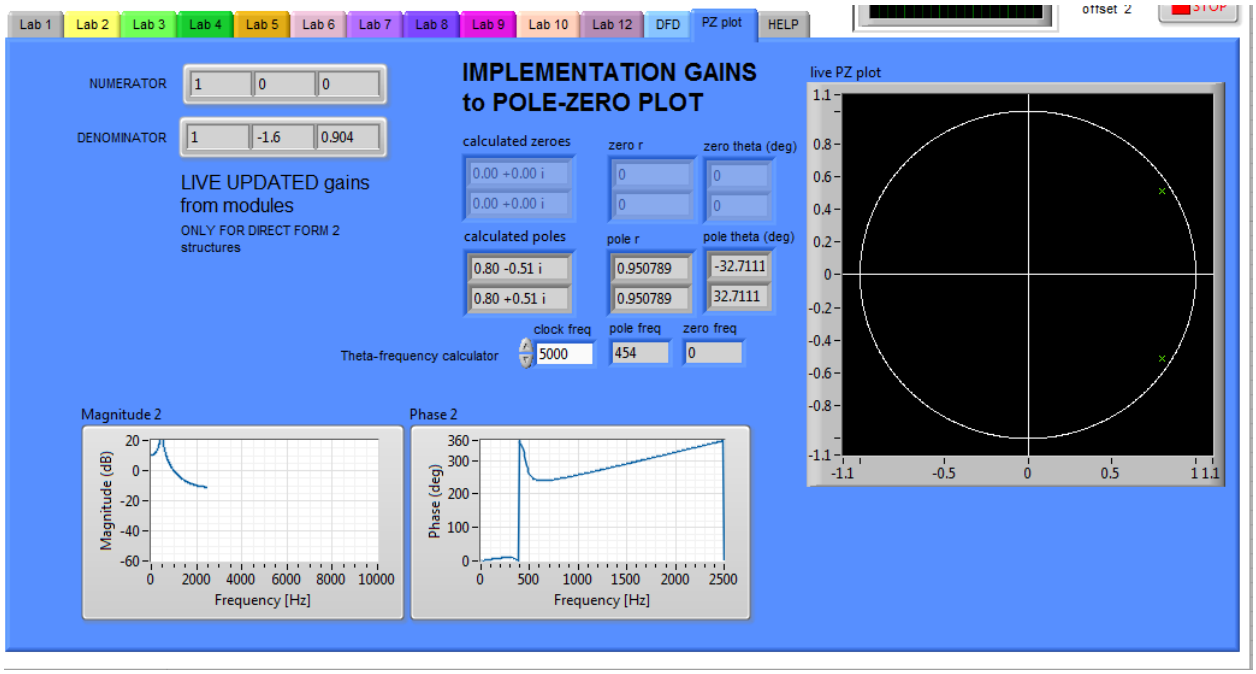


Figure 5: "PZ plot" TAB from SFP

Question 13
With a1 unchanged, gradually increase a_2 and observe the narrowing of the resonance. Continue until you see indications of unstable behaviour. At that point, remove the input signal and observe the output (if needed, increase a_2 a little more). Is it sinusoidal? Measure and record its frequency. Measure a_2. Calculate and plot the pole positions. Note especially whether they are inside or outside the unit circle.

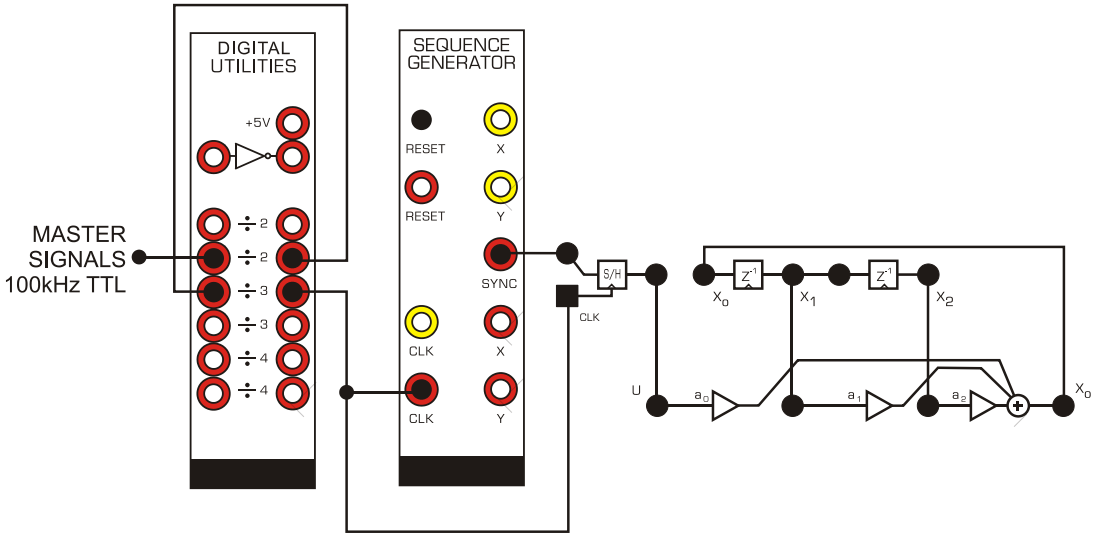


Figure 6: patching diagram for impulse response

2. We will now repeat step 11 in the time domain. Use the DIGITAL UTILITIES as a clock source and SEQUENCE GENERATOR to set up a unit pulse input. The SYNC signal from the SEQUENCE GENERATOR will act as a repetitive unit pulse source. What matters is that the unit pulses are far enough apart that each pulse is a unique event to the system under investigation.

Setting are as follows:
CLK: 16.6kHz
SEQUENCE GENERATOR: DIPS set to **UP:DOWN (medium sequence)**
ADDER gains: a0=1.0; a1=1.6; a2=-0.902;

Question 14
Begin with a_2 around -0.9. Describe the effect on the response as the magnitude of a_2 reduces. Measure the frequency of the oscillatory tail of the response and compare with your observations in step 5. Vary a_2 and observe the effects on the impulse response.

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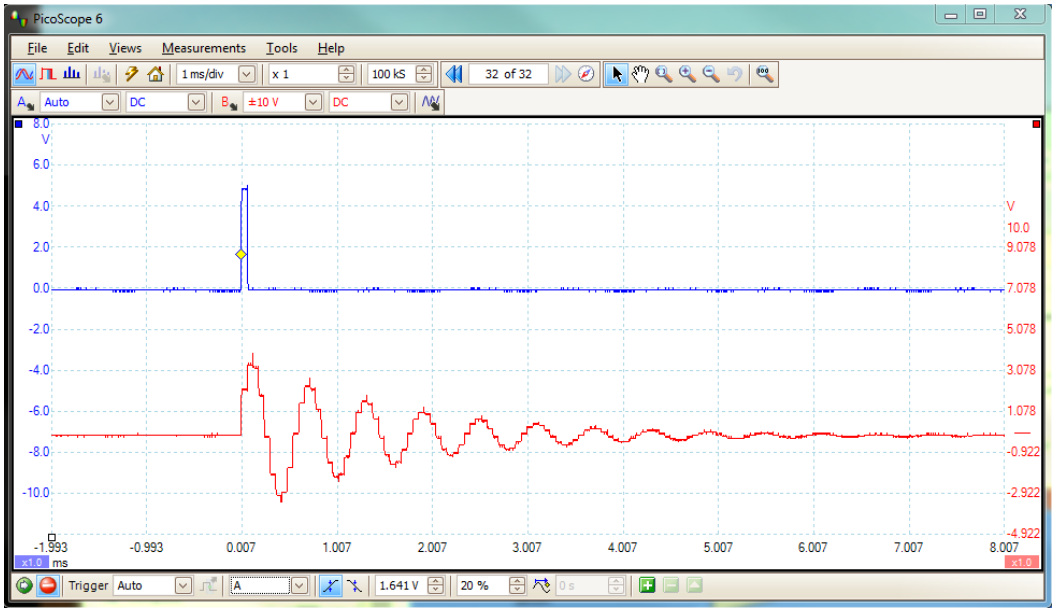


Figure 7: example of a typical pulse response for the default settings

Graph 4:pole only response plot



Figure 8: Setup with feedforward and feedback sections implemented, as per Figure 9 below

Part 2 - IIR with feedforward: second-order filters

In this part we implement and investigate the system in Fig 9. Note that the system with feedforward simply builds upon the previous system with feedback only. It also provides a new output point. The system response x0 for the feedback only, all-pole system is still available as a subset within this new arrangement and is unchanged by the additional feedforward elements. The feedforward elements simply add numerator terms to the overall transfer function which becomes y/u.

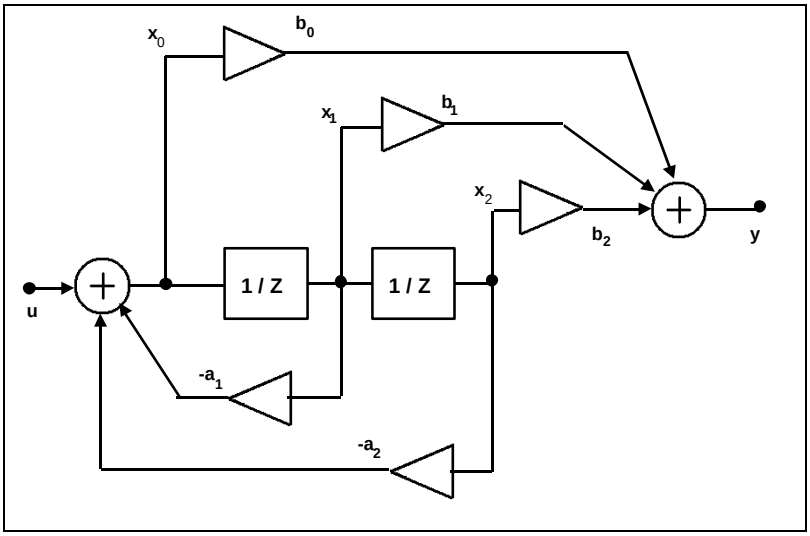


Figure 9: block diagram of 2nd-order Direct-form 2 structure with feedforward.

3. Use ADDER B in a z-TRANSFORM module to convert the model of Part 1 to the system in F9

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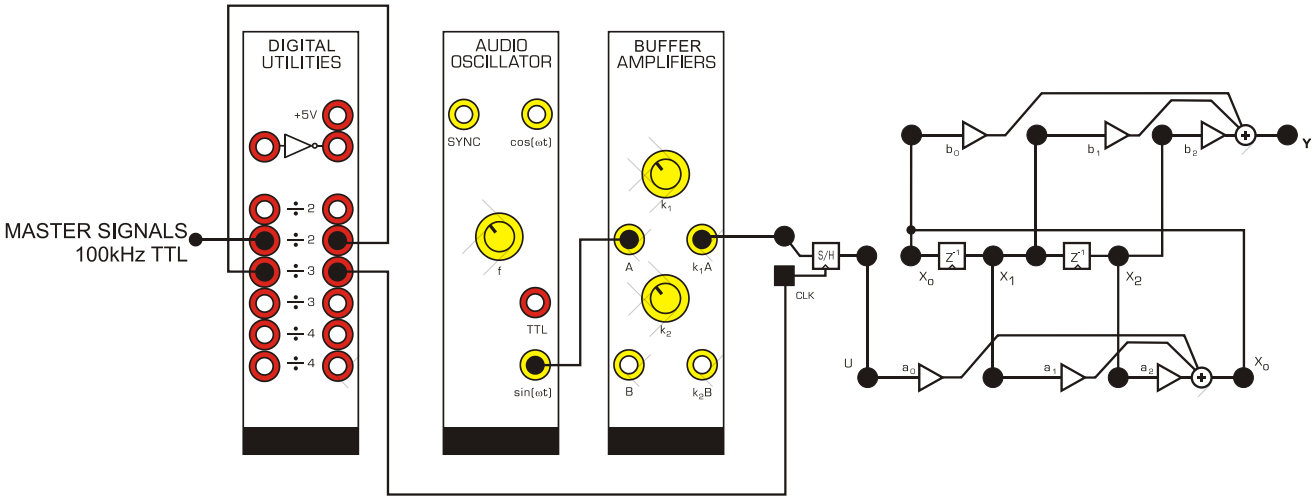


Figure 10: patching diagram for block diagram above

4. Implement case (i) in Prep (Q7).

Settings are as follows:

ADDER GAINS: b0=1; b1=2; b2=1; a0=1; a1=+1.6; a2= -0.902

CLK: 16.6kHz

AUDIO OSCILLATOR: Sinewave selected, FREQUENCY=1k; Amplitude (after BUFFERS)= 1V pp

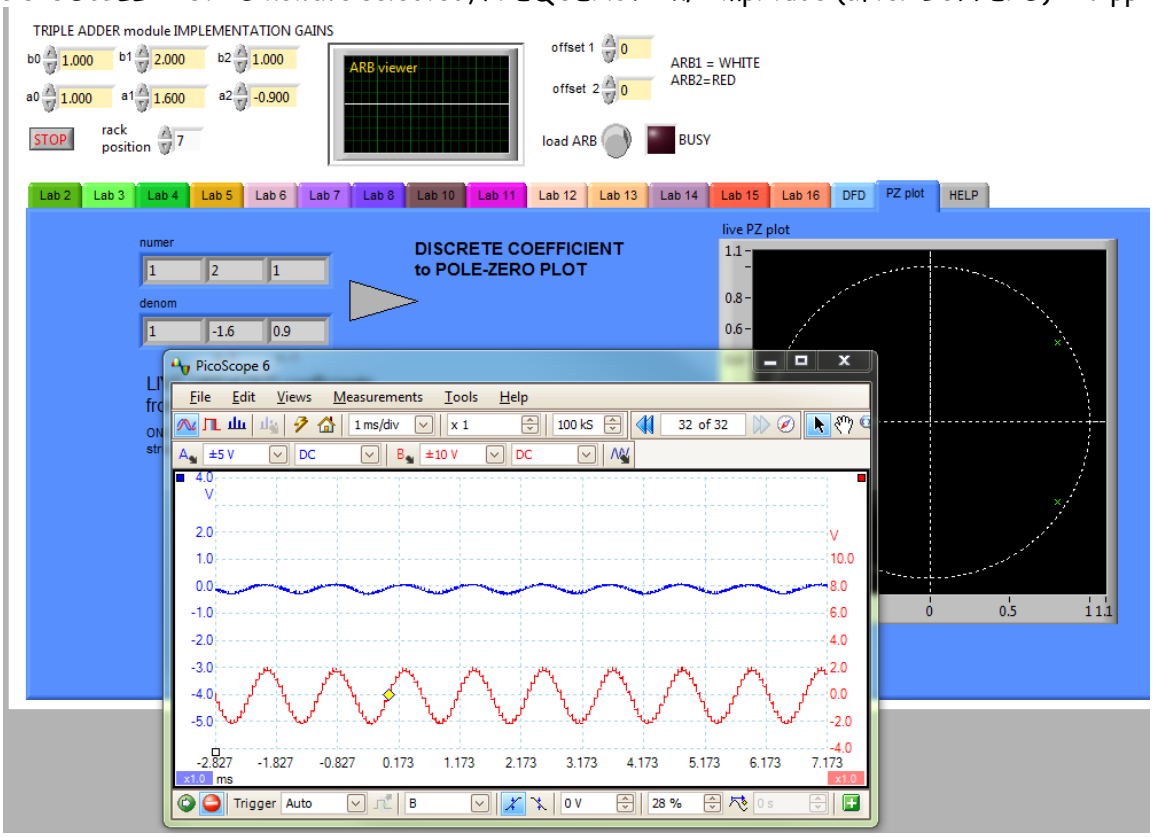


Figure 11: Using the Function Generator to sweep a sinusoid across the spectrum of interest.

Observation: the high pass band gain due to the selection of coefficients resulting in poles very close to the unit circle, as shown in the figure below.

NOTE: The poles are very close to the unit circle. In fact, the pole radius is 0.95. Hence the gain close to the poles is very large. You can use the PZ PLOT to visualize the poles and zeros for any "live" coefficient settings. Zeros are also present at z = -1.

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Measure the magnitude response |y/u| and confirm that it is a lowpass filter.

Graph 5: plot of responses

Question 15

In the model of step 14, adjust a_2 to reduce the peaking to a minimum. As well you will need to reduce the amplitude of the input signal to 0.5Vpp to reduce saturation. Confirm this for yourself. Plot the resulting response and measure the new value of a_2. Calculate and plot the new poles. Obtain an estimate of the theoretical magnitude response with these poles and compare this with the measured curve. Why was a_2 used for this rather than a_1?

Question 16

Change the polarity of b_1 in the lowpass of step 19 and show that this produces a highpass. Compare with your findings in Question 7.

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NOTE: the following three questions refer to the transfer function coefficient values. Remember to <u>negate the a1 and a2</u> values when settings them up as IMPLEMENTATION GAINS.

Question 17

Repeat for case (iii) in Question 7, that is with GAINS: $b_0 = 1$, $b_1 = 0$; $b_2 = -1$; $a_0 = 1$; $a_1 = -1.6$; $a_2 = 0.902$; Confirm this is a bandpass filter. Tune a_1 and a_2 to obtain a peak at 3.1 kHz and 3dB bandwidth 500Hz. Measure the resulting a_1 and a_2 and plot the new poles. Compare this with your findings in Question 7.

Question 18

Implement the following case: coefficients are $b_0 = 0.8$, $b_1 = 0$, $b_2 = 1$; $a_0 = 1$, $a_1 = 0$, $a_2 = 0.8$,. Note that $b_0 = a_2$ and $b_1 = a_1$. Measure the magnitude response. Confirm it is allpass. Locate the positions of the poles and zeros. Plot them below for your records.

Question 19

Change a_1 and b_1 coefficients to - 1.6 and confirm the response is still allpass. Examine the behaviour of the phase response. Look for the frequency of most rapid phase variation, and confirm this occurs near a pole.

Plot the poles and zeros below for your records.

Viewing spectrum of system with broadband noise input & FFT

As well as sweeping a single frequency signal from the AUDIO OSCILLATOR across the spectrum of interest it is also convenient to input a broad range of frequencies at once and view the overall output frequency response of the system. Creating a broadband analog noise signal was covered in Experiment 9. That methodology is shown in the figures below. You can revisit this experiment with this setup in place and see the relationships of poles and zeros to system response in real time.

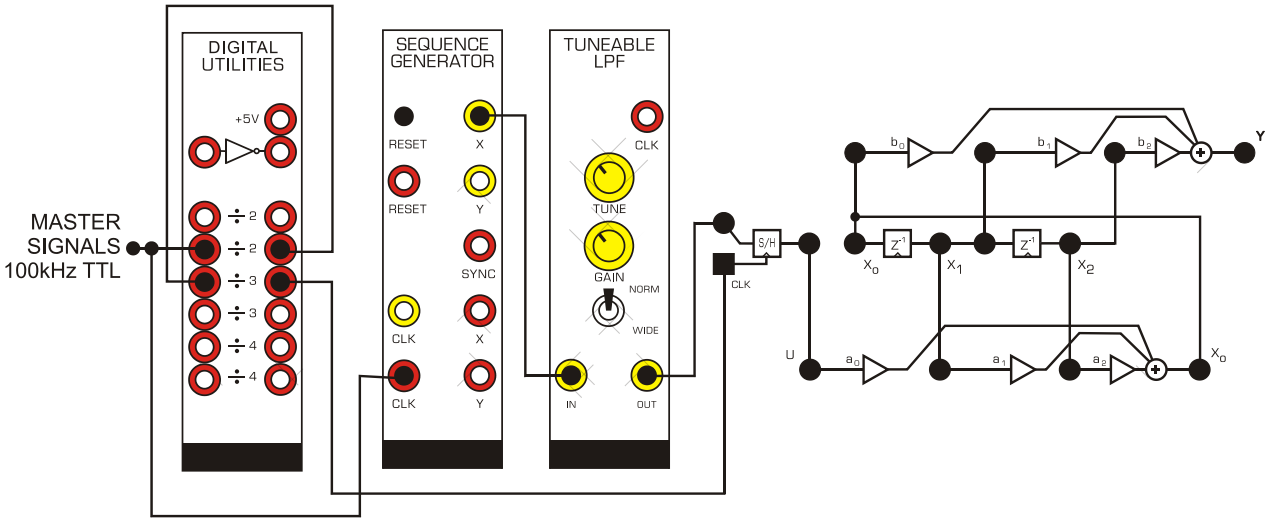


Figure 12: IIR filter with flat noise input

5. Ensure that **SEQUENCE GENERATOR DIP switches are set to positions DOWN:DOWN for the long sequence.** Set TLPF knobs to fully clockwise for now.
Set up the coefficients as per step 14.

Setting up the input noise signal:

6. i) Reduce the TLPF GAIN by rotating counter clockwise until the output Ch1 signal (red) is no longer saturated ie: less than 12V peak.
ii) Reduce the noise bandwidth to around 4khz by rotating the TLPF block's "Fc" control-counter clockwise. View the noise spectrum as the white trace on the SCOPE & FFT windows.

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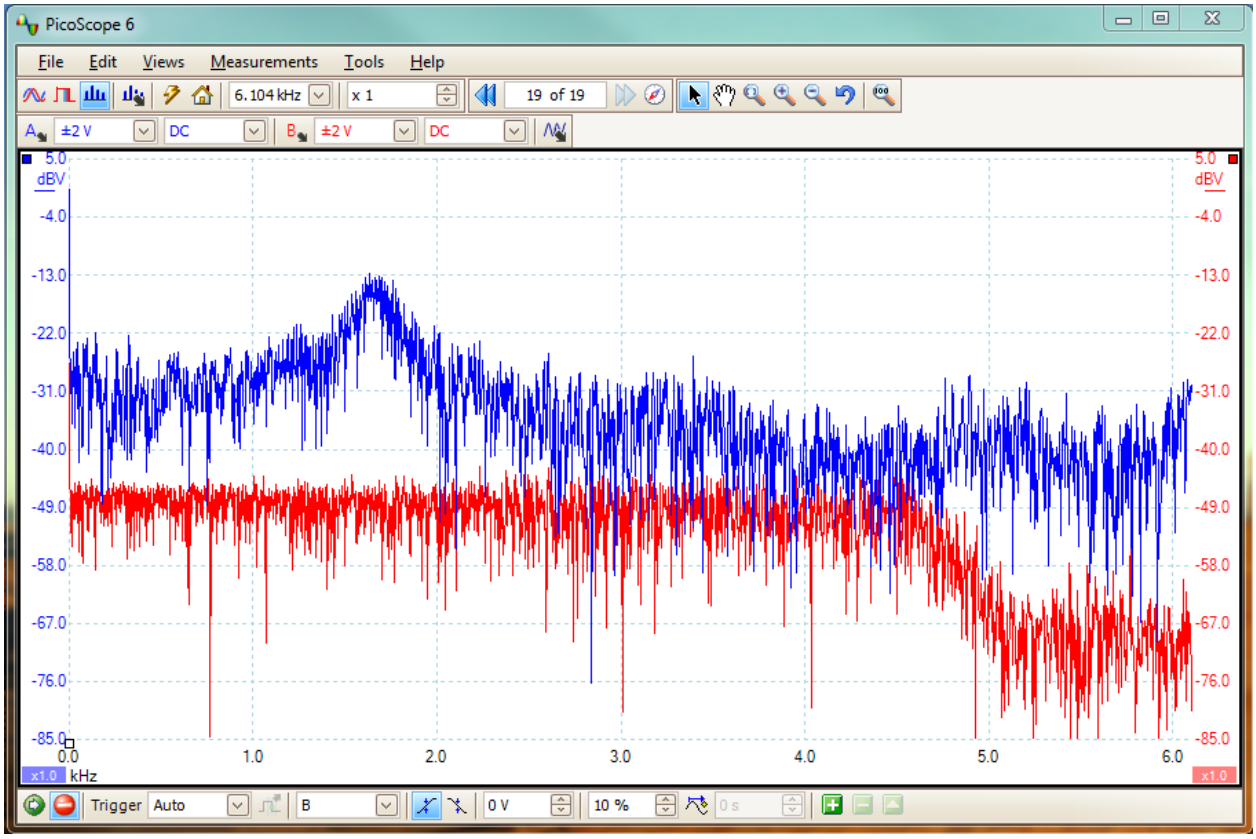


Figure 13: Example display of input noise and filter output spectrum

Due to high gain, input noise level is very small. Note the limited bandwidth of the input noise to maintain a flat input response. (SEQUENCE GENERATOR must be set to long sequence.)

At this point we can see and explore the issues relating to :
-controlling our input signal level and bandwidth
-viewing the response in both time and frequency domains
-setting up a transfer function with appropriate internal gains

We can also confirm that the peak of the response is correct according to the position of the poles. ie: PZ PLOT tells us that poles are at 32 degrees, hence we expect a peak close to 32/360*16,666 = 1500 Hz.

Question 20
Show your calculation of the where you expect the peak frequency to be using the pole position and sampling frequency.

Dynamically varying the poles and zeros to adjust response

7. View the frequency response while slowly varying the value of a_1. You will find that the peak frequency changes.

8. Find a range of a_1 settings that work well and then view PZ PLOT while varying across that range. You will see the poles moving and reflecting the changing **a_1** coefficient. (Theory states that $a_1=-2\sigma$, which is the real part of the pole and its conjugate.)

Question 21
Confirm this relationship from values displayed on PZ PLOT and show your working here:

9. Set a_1 back to +1.6

Question 22
Varying a_2 will vary the gain or peak level of the filter. Notice what happens in the time domain when a_2 = -1.0. The filter breaks into oscillation. View the poles again using PZ PLOT while varying a_2. (Theory states that $a_2 = r^2$).

10. Set a_2 back to -0.9.

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Using Digital Filter Design toolkit to implement filters

11. Switch to the DFD TAB. This will allow you to automatically load the computed coefficients for the selected filter into the implementation gains of the TRIPLE ADDER module. You can see the values on the TAB setup in the SFP.
NOTE: Maintain the order of your filter structure <= 2, to match the structure you have built.

12. Connect CH0 to " x_0 ", and CH1 to "Y" and view the internal signal levels. Set the TLPF GAIN higher but avoid saturating. These filters have lower internal gains than the previous ones. Vary the filter design type (at the DFD tab by clicking on "DESIGN METHOD" to select) and view the output responses using ZOOM FFT.

NOTE: You wil get an error message when changing between filter types, as the pass and stop settings will temporarily be invalid until you have completed the design specification setup.

Press the DFD "LOAD GAINS" button to transfer co-efficients to the IMPLEMENTATION GAINS control boxes when ready to implement.

You can expect to see an FFT display like so:

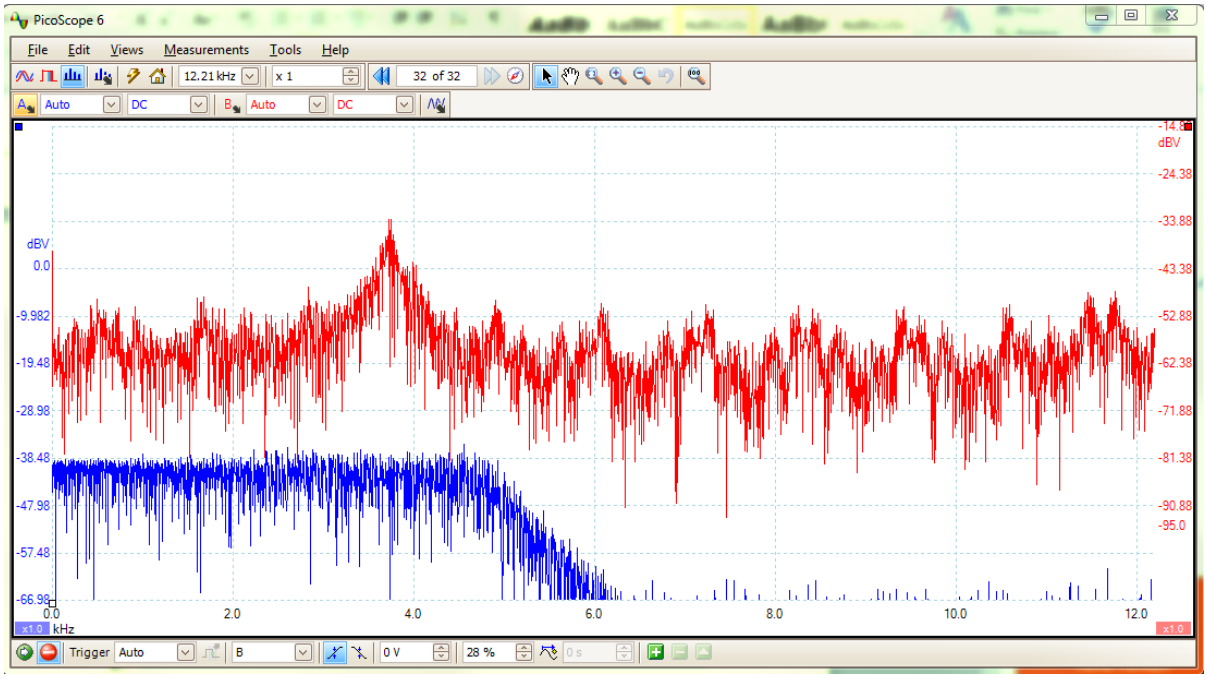


Figure 14: 2nd order bandpass filter response using FFT;

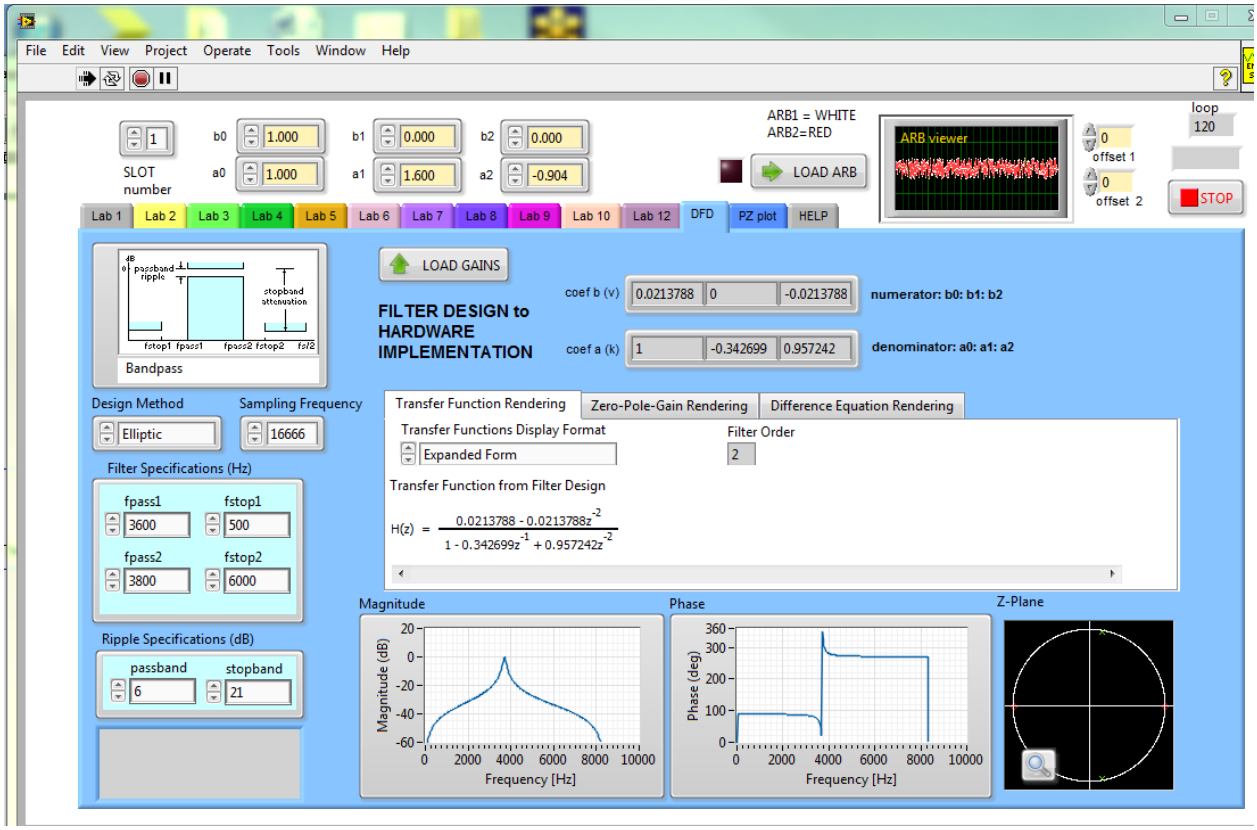


Figure 15: DFD TAB used to design filters and setup coefficients to the patched up filter

Question 23

Confirm that the hardware performs as designed by theory in terms of peak positions etc. You will have to use the pole positions mostly in these cases. Why ?

Question 24

Try varying design values and take note of the ORDER of the filter designed. NOTE that the experiment we have implemented can only support a 2nd order structure. Note your observations.

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Remember to be systematic in your patching together the experiment described above. Once completed, the arrangement will not need to be modified as all settings of the co-efficients will be via the SFP.

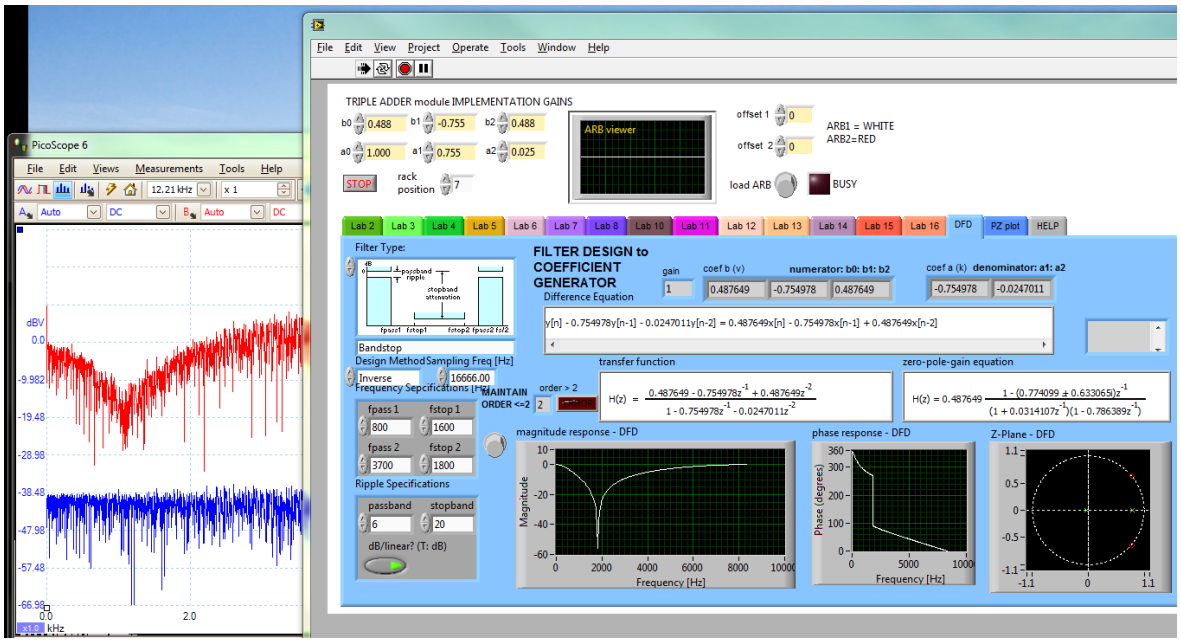


Fig 18: example of 2nd order bandstop structure, design (SFP) and implementation (PICOSCOPE)

Tutorial Questions

Q1. *Why do the complex poles and zeros occur in conjugate pairs in the cases covered in this lab?*

Q2. *Why is polar notation for complex valued poles and zeros preferred in the discrete-time context? Using examples from the lab, explain the importance of the position of complex poles/zeros relative to the unit circle when estimating frequency responses.*

Q3. *Re Eqn 3, keeping a_1 constant, plot the locus of the upper half plane pole with respect to a_2. Do this for several suitable values of a_1. Holding a_2 constant, repeat this with respect to a_1. Use the resulting contours to explain your observations in Q11 and Q12 .*

Q4. *Determine the conditions on a_1 and a_2 for the poles to be complex. Display this graphically on a plane (i.e. with a_2 = 0 as horizontal axis and a_1=0 as vertical axis).*

Q5. *Calculate and plot the poles and zeros in Q19. Satisfy yourself that they share the same radial line. Show that $z_1 = 1/p_1^*$.*

Q6. *Prove that the values of the coefficients in Q18 and Q19 generate a constant magnitude response over all frequencies. Write down the coefficient relationship in the transfer function of a fourth-order allpass.*

Q7. *Consider the unit pulse response in Step 12. What is the effect on the decay rate as the bandwidth is decreased? Find a simple formula or rule of thumb to express this relationship.*

Q8. *Show that the magnitude responses at nodes x_1 and at x_2 are the same as at x_0 (can be demonstrated without math).*

Q9. *Consider a bandpass filter realized with a_2 = 0.98. What is the maximum deviation allowable in a_2 to maintain a bandwidth tolerance of 5 percent?*

Q10. *Consider the following assertion: "Continuous-time filters can be considered as a limiting case of discrete-time filters, as the sampling frequency to bandwidth ratio gets very large". Hint: show that the poles and zeros migrate to the area near (1,0) as the Nyquist ratio increases and compare the shapes of the unit circle and of the j axis in that region.*

Q11. *Find out the meaning of the term "maximally flat". Is this description applicable to the filter produced in Q15 by reducing the value of a_2?*

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